数理解析研究所講究録
1090 巻 1999 年 126-130



# Formal Microlocalization

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**Formal Specialization and Asymptotic Expansions** (see [C])

First, let $X$ be a real analytic manifold with $\mathcal{C}_X^\infty$ the sheaf of complex valued $C^\infty$-functions on $X$. Let $U$ be a subanalytic open subset and $Z$ a subanalytic closed subset of $X$. Recall that the Whitney functor, $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^\infty$, defined by Kashiwara and Schapira [KS1], is characterized by:

$$\begin{array}{rcl} \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X) & \to & \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X) \\ \mathbb{C}_U & \mapsto & \mathcal{I}^\infty_{X,X\setminus U}, \text{ the subsheaf of } \mathcal{C}_X^\infty \text{ consisting of sections vanishing} \\ & & \text{at infinite order on } X\setminus U, \\ \mathbb{C}_Z & \mapsto & \text{the sheaf of complex valued functions } C^\infty \text{ on } Z \text{ in the sens} \\ & & \text{of Whitney.} \end{array}$$

Recall that a $C^\infty$-function on a subset $A$ of $\mathbb{R}^n$ in the sense of Whitney (see [W]) is a family $F = (F^k)_{k\in\mathbb{N}^n}$ of continous functions on $A$ such that:
$\forall m \in \mathbb{N}$, $\forall k \in \mathbb{N}^n, |k| \leqslant m$, $\forall x \in A$, $\forall \varepsilon > 0$, there is a neighborhood $U$ of $x$ such that $\forall y, z \in U \cap A$

$$\left| F^k(z) - \sum_{|j+k|\leqslant m} \frac{(z-y)^j}{j!} F^{j+k}(y) \right| \leqslant \varepsilon . dist(y,z)^{m-|k|} \tag{1}$$

Let $M$ be a submanifold of $X$, and $\tilde{X}$ the normal deformation of $X$ along $M$. We follow the notations of [KS2]:

$$\begin{tikzcd} T_M X \arrow[r, hook, "s"] \arrow[d, "\tau"] & \tilde{X} \arrow[d, "p"] & \Omega = \{t>0\} \arrow[l, hook'] \\ M \arrow[r, "i"] & X & \end{tikzcd}$$

**Definition 1.** – Let $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$. We set:

$$w\nu_M(F, \mathcal{C}_X^\infty) = s^{-1} R\mathcal{H}om_{\mathcal{D}_{\tilde{X}}}(\mathcal{D}_{\tilde{X}\to X}, (p^{-1}F)_{\overline{\Omega}} \overset{\mathrm{w}}{\otimes} \mathcal{C}_{\tilde{X}}^\infty), \tag{2}$$

and call it the Whitney specialization of $F$ along $M$.

From now on, $X$ will be a complex analytic manifold. We denote by $\overline{X}$ the complex conjugate of $X$, and by $X_{\mathbb{R}}$ the real underlying manifold. Recall the definition of the functor of formal cohomology: $F \overset{w}{\otimes} \mathcal{O}_X = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{w}{\otimes} \mathcal{C}^{\infty}_{X_{\mathbb{R}}})$.

**Definition 2.** – Let $M$ be a submanifold of $X_{\mathbb{R}}$ and $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$. We set:

$$w\nu_M(F, \mathcal{O}_X) = R\mathcal{H}om_{\tau^{-1}i^{-1}\mathcal{D}_{\overline{X}}}(\tau^{-1}i^{-1}\mathcal{O}_{\overline{X}}, w\nu_M(F, \mathcal{C}^{\infty}_{X_{\mathbb{R}}})), \tag{3}$$

and call it the formal specialization of $F$ along $M$. If $F = \mathbb{C}_X$ or $F = \mathbb{C}_{X \setminus M}$, we denote it by $w\nu_M(\mathcal{O}_X)$ and $w^0\nu_M(\mathcal{O}_X)$, respectively.

**Proposition 3.** – *Let $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$ and $v \in T_M X$. Then:*

$$H^k(w\nu_M(F, \mathcal{O}_X))_v \simeq \varinjlim \mathrm{R}^k\Gamma(X; F_{\overline{U}} \overset{w}{\otimes} \mathcal{O}_X),$$

*where $U$ ranges through the family of subanalytic open subsets of $X$ such that $v \notin C_M(X \setminus U)$.*

Actually we obtain the asymptotic expansions. Let $U$ be an open subanalytic relatively compact regular contractible subset of $X$ and $f \in \mathcal{O}(U)$. We say that $f$ admits an asymptotic expansion in $U$ along $M$ if it verifies one of the three equivalent following propositions:

(*i*) $\nu_M f \in \Gamma(V(U); w\nu_M(\mathcal{O}_X))$,

(*ii*) For all proper subsector $U'$ of $U$, $f|_{U'}$ is extendible to a $C^{\infty}$-function on $X$,

(*iii*) In a local coordinates system $(z) = (x, y) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$, where $M$ is defined by $\{x = 0\}$, there is formal serie $\sum_k a_k(y)x^k$ with coefficients $\mathrm{C}^{\infty}$ in an neighborhood of $\overline{U} \cap M$ in $M$ such that for all proper subsectors $U'$ of $U$, and for all multi-indices $N \in \mathbb{N}^{n-p}$, there is a constant $C > 0$, such that

$$\forall z \in U', \quad \left| f(z) - \sum_{k<N} a_k(y)x^k \right| \leqslant C|x^N|.$$

Then considering the distinguished triangle :

$$w^0\nu_M(\mathcal{O}_X) \to w\nu_M(\mathcal{O}_X) \to \mathbb{C}_M \overset{w}{\otimes} \mathcal{O}_X \xrightarrow{+1}$$

with $X = \mathbb{C}$ and $M = \{0\}$ we reobtain a result of Malgrange [Mal] and Sibuya [Si].

Unfortunately we have yet construct a formal specialization along an analytic subset. But using the method we hope to find the strongly asymptotically developable function of Majima [Maj]. In that case, the equivalence of (*ii*) and (*iii*) is already proved by Zurro [Z].

**Formal Microlocalization**

Let us consider the inverse Fourier-Sato transform of the formal specialization. We denote by $p_1$ and $p_2$ the first and second projections from $T_M X \underset{M}{\times} T_M^* X$. Let $P = \{(x, y) \in T_M X \times_M T_M^* X / < x, y > \geqslant 0\}$.

$$\begin{tikzcd}[column sep=small]
P \arrow[r, hook] & T_M X \underset{M}{\times} T_M^* X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
T_M X \arrow[dr, "\tau"'] & & T_M^* X \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

**Definition 4.** – We set:

$$w\mu_M(F, \mathcal{O}_X) = (w\nu_M(F, \mathcal{O}_X))^{\vee} = Rp_{2!}(p_1^! w\nu_M(F, \mathcal{O}_X))_P \tag{4}$$

and call it the formal microlocalization of $F$ along $M$.

**Proposition 5.** – *Let* $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$ *and* $p \in \dot{T}_M^* X$. *Then:*

$$H^k(w\mu_M(F, \mathcal{O}_X))_p \simeq \varinjlim_{U} H^{k+l}(F_U \overset{w}{\otimes} \mathcal{O}_X)_{\pi(p)} \tag{5}$$

*where* $U$ *ranges through the family of subanalytic open subsets of* $X$ *such that* $p \in int(C_M(U)^{oa})$, *the interior of the polar set to* $U$ *and* $l$ *is the codimension of* $M$ *in* $X$.

We denote by $\Delta$ the diagonal of $X \times X$. Let $q_1$ and $q_2$ the first and second projections defined on $X \times X$, Let us identify $T^*_{\Delta}(X \times X)$ with $T^* X$ by the first projection from $T^*(X \times X) = T^* X \times T^* X$.

**Definition 6.** – Let $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$. We set:

$$F \overset{w}{\underset{\mu}{\otimes}} \mathcal{O}_X = R\mathcal{H}om_{\pi^{-1}\mathcal{D}_{X\times X}}(\pi^{-1}\mathcal{D}_{X\times X \xrightarrow{q_1} X}, w\mu_{\Delta}(q_2^{-1}F, \mathcal{O}_{X\times X})). \tag{6}$$

**Proposition 7.** – *Let* $M$ *be a submanifold of* $X_{\mathbb{R}}$ *and* $k$ *the immersion of* $T_M^* X$ *in* $T^* X$. *Then:*

$$\mathbb{C}_M \overset{w}{\underset{\mu}{\otimes}} \mathcal{O}_X \simeq k_* w\mu_M(\mathbb{C}_X, \mathcal{O}_X) \tag{7}$$

**Proposition 8.** – *Let* $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$ *and* $p \in \dot{T}^* X$. *Then:*

$$H^k(F \overset{w}{\underset{\mu}{\otimes}} \mathcal{O}_X)_p \simeq \varinjlim_{U,V} H^k(Rq_{1!}(q_2^!(F_{\overline{U}}))_V \overset{w}{\otimes} \mathcal{O}_X)_{\pi(p)} \tag{8}$$

*where* $U$ *ranges through the family of neighborhood of* $\pi(p)$ *in* $X$ *and* $V$ *ranges through the family of subanalytic open subsets of* $X \times X$ *such that* $(p^a, p) \in int(C_{\Delta}(V)^{oa})$.

**Proposition 9.** – *Let* $F \in \mathrm{Ob}(\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X))$. *Then:*

$$R\pi_*(F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X) \simeq F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X, \tag{9}$$

$$R\pi_!(F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X) \simeq F \otimes \mathcal{O}_X, \tag{10}$$

*and we have the distinguished triangle:*

$$F \otimes \mathcal{O}_X \to F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \to R\dot{\pi}_*(F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X|_{\dot{T}^*X}) \xrightarrow{+1}. \tag{11}$$

Let $X$ and $Y$ be two complex manifolds. We denote by $p_X$ and $p_Y$ the first and second projection from $T^*(X \times Y)$ to $T^*X$ and $T^*Y$. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\pi^{-1}\mathcal{D}_{X\times Y})$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\pi^{-1}\mathcal{D}_Y)$, $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ and $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y})$. We denote :

$$\begin{aligned} \mathcal{M} \underset{\mathcal{D}_Y}{\circ} \mathcal{N} &= Rp_{X!}(\mathcal{M} \underset{\mathcal{D}_Y}{\otimes} p_Y^{a\,-1}\mathcal{N}) \\ K \circ F &= Rq_{Y!}(K \otimes q_Y^{-1}F). \end{aligned}$$

Using the morphism constructed in the last chapter of [KS1]:

$$\mathcal{T}hom(F, \mathcal{O}_X) \underset{\mathcal{O}_X}{\otimes} (F \otimes G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \to G \overset{\mathrm{w}}{\otimes} \mathcal{O}_X,$$

we obtain a morphism for integral transformation with the functor $\mathcal{T}\mu hom$ defined in [A].

**Theorem 10.** – *Let* $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ *and* $K \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{X\times Y})$. *Then we have a natural morphism:*

$$\mathcal{T}\mu hom(K, \mathcal{O}^{(0,d_Y)}_{X\times Y}[d_Y])^a \underset{\mathcal{D}_Y}{\circ} ((K \circ F) \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_Y) \to F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X. \tag{12}$$

From this theorem, with $X = Y$ and $K = \mathbb{C}_\Delta$, we get a natural morphism:

$$(\mathcal{E}^{\mathbb{R},f}_X)^a \otimes (F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X) \to F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X.$$

In particular, $H^j(F \overset{\mathrm{w}}{\underset{\mu}{\otimes}} \mathcal{O}_X)_p$ has a structure of $(\mathcal{E}_X)_p$-module for any $p$ in $T^*X$.

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